Fermion mass hierarchy and non-hierarchical mass ratios in $\mathbf{SU(5)\times U(1)_F}$

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We consider a $SU(5)\times U(1)_F$ GUT-flavor model in which the number of effects that determine the charged fermions Yukawa matrices is much larger than the number of observables, resulting in a hierarchical fermion spectrum with no particular regularities. The GUT-flavor symmetry is broken by flavons in the adjoint of $SU(5)$, realizing a variant of the Froggatt-Nielsen mechanism that gives rise to a large number of effective operators. By assuming a common mass for the heavy fields and universality of the fundamental Yukawa couplings, we reduce the number of free parameters to one. The observed fermion mass spectrum is reproduced thanks to selection rules that discriminate among various contributions. Bottom-tau Yukawa unification is preserved at leading order, but there is no unification for the first two families. Interestingly, $U(1)_F$ charges alone do not determine the hierarchy, and can only give upper bounds on the parametric suppression of the Yukawa operators.

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I. INTRODUCTION

The Standard Model (SM) provides and accurate description of particle physics phenomena. Particles interactions are derived from local symmetries and are explained at a fundamental level by the gauge principle. Myriads of experimental tests have confirmed the correctness of this picture. However, the SM cannot explain the values of the particle masses and mixing angles, and to get insight into this issue a new theory is required.

The mass spectrum of the charged fermions has a strong hierarchical structure that ranges over five orders of magnitude. Apart from this obvious feature, only a few regularities are observed, the most certain of which is that the bottom and tau masses converge towards a similar value when extrapolated to some large energy scale $\sim 10^{16}$ GeV. In the context of the MSSM, a recent analysis finds [1]

$$\frac{m_b}{m_\tau} = 1.00^{+0.04}_{-0.4}. \tag{1}$$

Supersymmetric unification of the three gauge couplings occurs at the same energy scale, and this hints to a grand unified theory (GUT) that can explain elegantly this result. It is thus likely that m_b-m_τ unification is not a numerical accident (although it could be only an approximate result [26]). For the down-quark and leptons of the first two generations unification does not work, but a different GUT relation exists: $3m_s/m_\mu = m_d/3m_e = 1$. This relation was suggested long ago by Georgi and Jarlskog (GJ) [2] that also showed how they could be obtained in the context of $SU(5)$ by means of a **45** Higgs representation. The GJ relations are less certain: the analysis in [1] quotes (for $\tan\beta = 1.3$ [27]):

$$\frac{3m_s}{m_\mu} = 0.70^{+0.8}_{-0.05}, \qquad \frac{m_d}{3m_e} = 0.82 \pm 0.07. \tag{2}$$

It is then likely that a more complicated mechanism is responsible for the values of these mass ratios. As regards the oldest mass matrix relation $V_{us} \approx \sqrt{m_d/m_s}$ that was proposed forty years ago by Gatto, Sartori and Tonin (GST) [3], it is still in good agreement with the experimental values [1]. A few other empirical relations were proposed in [4].

The absence of enough well established regularities in the fermion mass pattern leaves open the way to many different explanations of the origin of fermion masses, and is probably the main reason why, in spite of all the theoretical efforts, no compelling theory has yet emerged. Most theoretical efforts concentrated in reducing the number of fundamental parameters as much as possible, by imposing symmetries and/or by assuming special textures for the Yukawa matrices, like symmetric forms, or a certain number of zero elements (for reviews of different ideas see e.g. refs. [5, 6, 7, 8]). Clearly, a number of parameters smaller than the number of observables would yield some testable predictions, that can rule out some possibilities and favor others. Moreover, there is also the hope that a reduced set of parameters could reveal some regular pattern that could provide some hint of the correct theory.

However, it is also possible that the opposite situation is true. Namely, that the number of different effects that contribute to determine the values of the fermion masses is much larger than the number of observables. Then, even if the fundamental contributions are determined by some simple rule or symmetry principle, it is likely that in the fermion spectrum no regularities would appear. If this is the case, and if the scale of the related new physics is inaccessibly large, then identifying the correct solution to the fermion mass problem could be an impossible task.

In this paper we discuss a framework that realizes

this possibility. We assume the GUT-flavor symmetry $SU(5)\times U(1)_F$, that is broken down to the SM by $SU(5)$ adjoint Higgs representations charged under $U(1)_F$, that hence play also the role of flavon fields. The Froggatt-Nielsen (FN) mechanism [9] is incorporated with the variant that since the flavons are not singlets under $SU(5)$, the heavy vectorlike fields responsible for generating the effective mass operators for the quarks and leptons can belong to several different representations. To simplify things, and to highlight the special features of this framework, we assume that all the heavy states have the same mass, and that at the fundamental level the Yukawa couplings are universal. This yields a scheme with just one relevant parameter, that is the ratio between the vacuum expectation value (vev) of the flavons and the heavy fermions mass. This parameter is responsible for the fermion mass hierarchy, while the details of the spectrum are determined by several non-hierarchical (and computable) group theoretical coefficients, that depend on the way the heavy FN states are assigned to $SU(5)$ representations. As we will see, the number of contributions to the fermion mass operators, each one weighted by a different $SU(5)$ coefficient, overwhelms the number of observable, completely hiding the underlying $SU(5)$ symmetry.

II. THE GENERAL FRAMEWORK

We work in the framework of a supersymmetric GUT-flavor model based on the gauge group $SU(5)\times U(1)_F$ (supersymmetry is needed for $SU(5)$ to be a phenomenologically viable GUT). Different realizations of models based on $SU(5)\times U(1)_F$ have been proposed expecially for what concerns the implications for the possible patterns of neutrino masses and mixing angles (see [10] for a review and list of references). In the following we list the main ingredients and assumptions that underlie our framework.

$SU(5)$ Grand Unified Symmetry. In $SU(5)$ GUTs, the $SU(2)$ lepton doublets $L=(\nu,e)^T$ and the down-quark singlets d^c are assigned to the fundamental conjugate representation $\bar{\mathbf{5}}$, while the quark doublets $Q=(u,\,d)^T$, the up-type quark singlets u^c and the lepton singlets e^c fill up the two-index antisymmetric $\mathbf{10}$:

$$\bar{\mathbf{5}}=\begin{pmatrix} d^c\\ d^c\\ d^c\\ e\\ -\nu\end{pmatrix},\quad \mathbf{10}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & u^c & -u^c & u & d\\ -u^c & 0 & u^c & u & d\\ u^c & -u^c & 0 & u & d\\ -u & -u & -u & 0 & e^c\\ -d & -d & -d & -e^c & 0\end{pmatrix}. \quad (3)$$

The Higgs field ϕ_d responsible for the down-quarks and lepton masses belong to another $\bar{\mathbf{5}}^{\phi_d}$ with $\langle\bar{\mathbf{5}}^{\phi_d}\rangle\sim\mathrm{diag}(0,0,0,0,-v_d)$, while ϕ_u responsible for the masses of the up-quarks is assigned to a fundamental $\mathbf{5}_{\phi_u}$ with $\langle\mathbf{5}_{\phi_u}\rangle\sim\mathrm{diag}(0,0,0,0,v_u)$. As usual we define $\tan\beta\equiv v_u/v_d$. The Yukawa superpotential is

$$W_Y=\sqrt{2}\,Y^D_{IJ}\bar{\mathbf{5}}_{Ia}\mathbf{10}^{ab}_J\,\bar{\mathbf{5}}^{\phi_d}_b+\frac{1}{4}Y^U_{IJ}\epsilon_{abcde}\mathbf{10}^{ab}_I\mathbf{10}^{cd}_J\,\mathbf{5}^e_{\phi_u}, \quad (4)$$

where Y^D (Y^U) is the Yukawa couplings matrix for the down-quarks and leptons (up-quarks), $I\,,J=1,2,3$ are generation indices, $a,b,\dots=1,\dots,5$ are $SU(5)$ indices, and ϵ_{abcde} is the $SU(5)$ totally antisymmetric tensor.

One problem of $SU(5)$ GUTs is how to guarantee that the two electroweak Higgs doublets H_u and $\bar{H}_d$ contained respectively in $\mathbf{5}_{\phi_u}$ and $\bar{\mathbf{5}}^{\phi_d}$ remain light, while the color triplet components acquire a mass large enough to suppress proton decay below the experimental limits. The technical solution adopted in minimal $SU(5)$ is to invoke a fine tuned cancellation between a trilinear term coupling $\mathbf{5}_{\phi_u}$ and $\bar{\mathbf{5}}^{\phi_d}$ with the adjoint Σ (with vev $\langle\Sigma\rangle=V\,\mathrm{diag}(2,2,2,-3,-3)$) and an invariant mass term:

$$W_\phi\sim\bar{\mathbf{5}}^{\phi_d}_a(\Sigma^a_b+3M_\phi\delta^a_b)\mathbf{5}^{\,b}_{\phi_u}\,. \quad (5)$$

By choosing $M_\phi=V$ with an accuracy of one part in 10^{14} the contribution to the SM Higgs doublets is of the order of 100 GeV, while the color triplets acquire a GUT scale mass.

$U(1)_F$ flavor symmetry and FN mechanism. Two qualitative features are apparent in the (GUT scale) charged fermion mass spectrum: *i*) the structure is strongly hierarchical; *ii*) there is no obvious inter-family multiplet structure. The first feature hints to a spontaneously broken flavor symmetry in which the hierarchical structure is determined by powers of a small order parameter, while the second feature suggests that the flavor symmetry is likely to contain an Abelian factor [11].

The approach proposed long ago by Froggatt and Nielsen [9] realizes these two conditions. The basic ingredient is an Abelian flavor symmetry that forbids at the renormalizable level most of the fermion Yukawa couplings. The symmetry is spontaneously broken by the vev of a SM singlet flavon field $\langle S\rangle$. After the symmetry is broken a set of effective operators arises, that couple the SM fermions to the electroweak Higgs boson(s), and that are induced by heavy vectorlike fields with mass $M>\langle S\rangle$. The hierarchy of fermion masses results from the dimensional hierarchy among the various higher order operators that are suppressed by powers of the ratio $\langle S\rangle/M<1$. In turn, the suppression powers are determined by the Abelian charges assigned to the fermion fields. This mechanism has been thoroughly studied in different contexts like the supersymmetric SM [11], in frameworks where the horizontal symmetry is promoted to a gauge symmetry that can be anomalous [12, 13, 14] or non-anomalous [15], and with discrete Abelian symmetries [16]. When incorporated in the SM (or in the MSSM [11]) the FN mechanism allows to account qualitatively for the hierarchy in the fermion mass pattern and can also yield a couple of order-of-magnitude predictions. However, when applied to the simplest GUT models, like those based on $SU(5)$, the FN mechanism is less successful. On the one hand $U(1)_F$ breaking by $SU(5)$ singlet flavons does not account for the mass ratios m_s/m_μ, $m_d/m_e\neq 1$. On the other hand, the fact

that the five fermion multiplets for generation of the SM are reduced to just two $SU(5)$ representations eq. (3) implies much less freedom in choosing the $U(1)_F$ charges, and generally only the gross features of the fermion mass spectrum can be accounted for.

As we will discuss, these drawbacks can be overcome if the flavon fields are assigned to $SU(5)$ adjoint representations. Thus we assume that the same scalar multiplets that break $SU(5)$ down to the SM gauge group carry flavor charges and break also $U(1)_F$, playing effectively the role of the singlet flavon S in usual FN models. At each new order in the (small) symmetry breaking parameter a cascade of new effective operators appears. These operators are weighted by non-trivial $SU(5)$ group theoretical factors, and contribute differently to the down-quark and lepton mass matrices. This allows to explain the lifting of the mass degeneracy between leptons and quarks belonging to the same multiplet while, rather surprisingly, under certain conditions approximate b-τ unification can be preserved (see also ref. [17]).

Universal Yukawa interactions and heavy fermion masses. We implement the variant of the FN mechanism described above within a quite constrained framework, considering the possibility that the fermion mass structure could result from an underlying model in which at a high scale (larger than the GUT scale) the fundamental Yukawa couplings obey to some unification principle, analogous to the unification found for the gauge couplings. This assumption will be treated just as a constraining condition that, thanks to the large reduction in the number of free parameters, allows to highlight some general features of the model. In particular we will not speculate on the origin of this universality [28]. We will also assume a common mass ($M > \Lambda_{GUT}$) for all the heavy vectorlike representations (unlike the previous assumption, this condition can be implemented rather easily by assuming that the vectorlike masses are dominated by the common vev of a singlet scalar field). With these two assumptions there remains only one free parameter that is relevant to the problem, that is the dimensionless ratio between the symmetry breaking vev and the heavy mass M.

In summary, the scheme we are proposing embeds the FN explanation of the hierarchy of the fermion mass spectrum, but introduces additional group theoretical structures. They have a twofold effect: firstly they can change the naive hierarchy that one would infer from the $U(1)_F$ charge assignments (and could even produce texture zeros); secondly, they result in a large set of non hierarchical coefficients that depend on the field content of the model, and that can simulate rather well the absence of regular patterns in the effective Yukawa matrices. The important point is that these coefficients are computable, and we will illustrate their effects by evaluating the lepton and down-type quark Yukawa matrices up to third order in the small symmetry breaking parameter.

III. $U(1)_F$ CHARGE ASSIGNMENTS

In this section we discuss a set of conditions that can help us to identify the possible charge assignments for the SM fields.

A. Fermion mass hierarchy

A reasonable description of the charged fermion mass hierarchy can be given in terms of the following mass ratios (we assume from the start moderate values of $\tan\beta$):

$$m_{d,\,e} \,:\, m_{s,\,\mu} \,:\, m_{b,\,\tau} \;\approx\; \epsilon^3 \,:\, \epsilon^2 \,:\, \epsilon, \tag{6}$$

$$m_u \,:\, m_c \,:\, m_t \;\approx\; \epsilon^4 \,:\, \epsilon^2 \,:\, 1, \tag{7}$$

with $\epsilon \approx 1/20 - 1/30$. These relations imply that the order of magnitude of the determinants of the Yukawa matrices is

$$\det Y^U \sim \det Y^D \sim \epsilon^6. \tag{8}$$

We now assume that ϵ is related to the vev of flavon fields that, without loss of generality, carry a unit charge under the $U(1)_F$ symmetry. More precisely, we assume that the flavor symmetry is broken by scalar fields $\Sigma_{\pm 1}$ in the **24**-dimensional adjoint representation of $SU(5)$, where the subscripts ± 1 refer to the $U(1)_F$ charge values, and set the normalization for all the other charges. The vevs $\langle \Sigma_{+1} \rangle = \langle \Sigma_{-1} \rangle = V_a$ (as required by D flatness) with $V_a = V\,\mathrm{diag}(2,2,2,-3,-3)/\sqrt{60}$ are also responsible for breaking the GUT symmetry down to the electroweak-color gauge group. The order parameter for the flavor symmetry is then $\epsilon = V/M$ where M is the common mass of the heavy FN vectorlike fields. This symmetry breaking scheme has two important consequences:

1. Power suppressions in ϵ appear with coefficients related to the different entries in V_a, that distinguish the leptons from the quarks.
2. The FN fields are not restricted to the $\mathbf{5}, \bar{\mathbf{5}}$ or $\mathbf{10}, \bar{\mathbf{10}}$ multiplets as is the case when the $U(1)_F$ breaking is triggered by singlet flavons.

After the symmetry is broken, the effective Yukawa couplings generated for the charged fermions are suppressed *at least* as

$$Y^D_{IJ} \sim \epsilon^{|\bar{\mathbf{5}}_I + \mathbf{10}_J + \bar{\mathbf{5}}^{\phi_d}|}, \qquad Y^U_{IJ} \sim \epsilon^{|\mathbf{10}_I + \mathbf{10}_J + \mathbf{5}_{\phi_u}|}, \tag{9}$$

where we denote the F-charges with the same symbol than the multiplets, e.g. $F(\bar{\mathbf{5}}_I) = \bar{\mathbf{5}}_I$, $F(\mathbf{10}_I) = \mathbf{10}_I$, etc. Since we have two flavon multiplets $\Sigma_{\pm 1}$ with opposite charges, the horizontal symmetry allows for operators with charges of both signs, and hence the exponents in eq. (9) are absolute values of sum of charges. For models based on spontaneously broken family symmetries it

is natural to assume that the mass hierarchies (6) and (7) are determined by the diagonal terms in the Yukawa matrices, that is that the off-diagonal terms are smaller than the respective combinations of on-diagonal matrix elements. This gives for the multiplet charges the following conditions:

$$\begin{aligned}
&|\bar{\mathbf{5}}_3 + \mathbf{10}_3 + \bar{\mathbf{5}}^{\phi_d}| = 1, \qquad && |\mathbf{10}_3 + \mathbf{10}_3 + \mathbf{5}_{\phi_u}| = 0, \\
&|\bar{\mathbf{5}}_2 + \mathbf{10}_2 + \bar{\mathbf{5}}^{\phi_d}| = 2, \qquad && |\mathbf{10}_2 + \mathbf{10}_2 + \mathbf{5}_{\phi_u}| = 2, \\
&|\bar{\mathbf{5}}_1 + \mathbf{10}_1 + \bar{\mathbf{5}}^{\phi_d}| = 3, \qquad && |\mathbf{10}_1 + \mathbf{10}_1 + \mathbf{5}_{\phi_u}| = 4.
\end{aligned} \tag{10}$$

Since the top-quark mass is of the order of the electroweak breaking scale, Y^U_{33} must be the fundamental coupling of a renormalizable operator. Namely, the top Yukawa coupling must respect the flavor symmetry. All the other mass operators are not $U(1)_F$ invariant, and the corresponding Yukawa couplings are effective parameters suppressed by powers of ϵ.

B. Distinguishing fermion multiplets at the fundamental level.

When a pair of multiplets $\mathbf{r_1}$ and $\mathbf{r_2}$ are assigned to the same representation of the fundamental gauge group (in our case $SU(5)\times U(1)_F$), the gauge-interaction Lagrangian has a global $U(2)$ symmetry corresponding to rotations of $\mathbf{r_{1,2}}$. If also the Yukawa Lagrangian respects this global symmetry, one combination of the two multiplets decouples and remains massless. Since no massless fermions are observed, if this kind of symmetries exist, they must be broken. In the absence of any other fundamental 'label' that would distinguish $\mathbf{r_1}$ from $\mathbf{r_2}$, the only possibility is to introduce an explicit breaking. In Abelian models of flavor this breaking is usually provided *ad-hoc* by assuming that different $\mathcal{O}(1)$ coefficients multiply the Yukawa terms for $\mathbf{r_1}$ and $\mathbf{r_2}$. However, distinguishing $\mathbf{r_1}$ from $\mathbf{r_2}$ can be also considered as part of the flavor problem, and this part is left unexplained by the ad-hoc procedure. Of course one can assume that the different $\mathcal{O}(1)$ coefficient have an explanation at a more fundamental level, but this still implies that an additional (unspecified) structure able to distinguish $\mathbf{r_1}$ from $\mathbf{r_2}$ must exist. Therefore, one is forced either to give up the possibility of a full explanation of the flavor problem, or to assume that the model is incomplete at least in some parts.

Because of the assumption of universality of the fundamental Yukawa couplings, we cannot appeal to $\mathcal{O}(1)$ coefficients of unspecified origin, and accordingly we will require that each fermion multiplet is assigned to a different $SU(5)\times U(1)_F$ representation. Note that this conditions excludes the simple (and often used) charge assignments in which the hierarchical patterns (6), (7) are reproduced by assigning the same charge to all the $\bar{\mathbf{5}}_I$, and the hierarchy is determined by augmenting in each generation the charge of $\mathbf{10}_I$ by one unit (see [21] for a theoretical framework that could explain such a pattern of $U(1)_F$ charges).

C. Doublet-triplet splitting

Implementing the technical solution to the doublet-triplet splitting problem eq. (5) within our model implies two conditions. Firstly, to allow for the invariant mass term M_ϕ the charges of the multiplets containing the Higgs fields must be equal in magnitude and opposite in sign:

$$\bar{\mathbf{5}}^{\phi_d} + \mathbf{5}_{\phi_u} = 0. \tag{11}$$

Secondly, we need to introduce an adjoint Higgs representation Σ_0 neutral under $U(1)_F$ to implement the cancellation between the two contributions to the Higgs doublets mass. Note that replacing Σ in eq. (5) by an effective term $\Sigma_+\Sigma_-/M$ would suppress the masses of the color triplets by one power of ϵ, and this could imply an unacceptably fast proton decay. While our model does not shed any new light on the origin of the doublet-triplet splitting, the two conditions above must be imposed to ensure its (technical) consistency.

D. Charge assignments

The requirement that all the SM fermion multiplets are assigned to inequivalent $SU(5)\times U(1)_F$ representations together with the conditions eqs. (10) and (11) result in eight possible charge assignments that are listed in table I where, for simplicity, we have arbitrarily chosen vanishing charges for the Higgs fields $\bar{\mathbf{5}}^{\phi_d} = \mathbf{5}_{\phi_u} = 0$. (Reverting the sign of all charges gives trivially other eight possibilities.)

The effective Yukawa superpotential eq. (4) is invariant with respect to the following charge redefinitions:

$$\begin{aligned}
\bar{\mathbf{5}}_I &\rightarrow \bar{\mathbf{5}}_I \;+\; a; \\
\mathbf{10}_I &\rightarrow \mathbf{10}_I + b; \\
\bar{\mathbf{5}}^{\phi_d} &\rightarrow \bar{\mathbf{5}}^{\phi_d} - (a+b); \\
\mathbf{5}_{\phi_u} &\rightarrow \mathbf{5}_{\phi_u} - 2\,b,
\end{aligned} \tag{12}$$

with a and b two arbitrary real numbers. Under this redefinitions the charge of the Higgs bilinear term shifts as

$$F(\bar{\mathbf{5}}^{\phi_d}\mathbf{5}_{\phi_u}) \rightarrow F(\bar{\mathbf{5}}^{\phi_d}\mathbf{5}_{\phi_u}) - (a+3b), \tag{13}$$

and thus for $b = -a/3$ the redefinitions eq. (12) leave invariant also the condition eq. (11). Therefore, more precisely each row in table I identifies a one-parameter family of charges satisfying eqs. (10) and (11). The assignments in the last six rows (3)-(8) in the table must be discarded at once since they yield mass eigenvalues for the leptons and down-quarks of order $\sim v_d$. This is because in all these cases for some entries of the down and lepton Yukawa matrix Y^D the sum of the charges vanishes, and thus these entries are not suppressed by powers of ϵ. This does not happen for the first two assignments, that could lead to viable models.

	$\bar{\mathbf{5}}_1$	$\bar{\mathbf{5}}_2$	$\bar{\mathbf{5}}_3$	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$
(1)	-5	-3	$+1$	$+2$	$+1$	0
(2)	$+5$	-3	$+1$	-2	$+1$	0
(3)	$+1$	-3	-1	$+2$	$+1$	0
(4)	$+5$	-3	-1	-2	$+1$	0
(5)	-5	-3	-1	$+2$	$+1$	0
(6)	-1	-3	$+1$	-2	$+1$	0
(7)	-5	$+1$	-1	$+2$	$+1$	0
(8)	$+5$	$+1$	-1	-2	$+1$	0

TABLE I: The eight $U(1)_F$ charge assignments that satisfy the hierarchy conditions in eq. (10) together with eq. (11), and that label in a distinguishable way all the fermion multiplets. In all the cases $\bar{\mathbf{5}}^{\phi_d} = \mathbf{5}_{\phi_u} = 0$ has been chosen for simplicity.

E. Gauge anomalies

We assume that the $SU(5) \times U(1)_F$ symmetry is gauged, and thus it must be free of gauge anomalies. Anomaly cancellation yields three conditions corresponding to the vanishing of the gravitation-$U(1)_F$ anomaly, of the pure $U(1)_F^3$ anomaly, and of the mixed $SU(5)^2 \times U(1)$ anomaly. The first two anomalies can be canceled by adding $SU(5)$ singlet states with suitable $U(1)_F$ charges (e.g. two singlet 'neutrinos' [22]). The mixed anomaly can be canceled by invoking the Green-Schwarz mechanism [23], in which case the $U(1)_F$ symmetry is called *anomalous*, or by satisfying the condition

$$\mathcal{A} \equiv \mathcal{I}_{\mathbf{5}} \left(\bar{\mathbf{5}}^{\phi_d} + \mathbf{5}_{\phi_u} + \sum_{I=1}^{3} \bar{\mathbf{5}}_I \right) + \mathcal{I}_{\mathbf{10}} \sum_{I=1}^{3} \mathbf{10}_I = 0, \quad (14)$$

in which case the symmetry is *non-anomalous*. In eq. (14) $\mathcal{I}_{\mathbf{5}}$ and $\mathcal{I}_{\mathbf{10}}$ denote the index of the respective representations: $\mathrm{Tr}[\mathbf{r}^a \mathbf{r}^b] \equiv \mathcal{I}_{\mathbf{r}}\, \delta^{ab}$ ($\mathbf{r} = \mathbf{5}, \mathbf{10}$), while $\bar{\mathbf{5}}^{\phi_d}$, $\mathbf{5}_{\phi_u}$, $\bar{\mathbf{5}}_I$, $\mathbf{10}_I$ denote the $U(1)_F$ charges. Condition (11) implies that the Higgs representations $\bar{\mathbf{5}}^{\phi_d}$ and $\mathbf{5}_{\phi_u}$ do not contribute to $\mathcal{A}$. Then, under the charge redefinitions (12) restricted to $b = -a/3$ to preserve (11), the anomaly coefficient shifts as

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A} + 3a \left(\mathcal{I}_{\mathbf{5}} - \frac{\mathcal{I}_{\mathbf{10}}}{3} \right). \quad (15)$$

It is a numerical coincidence that the ratio of the indexes for the $\mathbf{10}$ and $\bar{\mathbf{5}}$ $SU(5)$ representations is $\mathcal{I}_{\mathbf{10}}/\mathcal{I}_{\mathbf{5}} = 3$ [24] implying that the remaining one parameter freedom in redefining the charges is ineffective for canceling the mixed anomalies. More precisely, no charge redefinition is possible that cancels the mixed anomaly and simultaneously preserves eq. (11). By inspecting the first two rows in table I we can then conclude that the assignments in (1) correspond to a family of anomalous models, and those in (2) to a family of anomaly free models.

The one parameter freedom in charge redefinition can still be useful to forbid all the trilinear operators $\bar{\mathbf{5}}_I \bar{\mathbf{5}}_J \mathbf{10}_K$ that violate baryon and lepton number, as well as the lepton number violating bilinear terms $\bar{\mathbf{5}}_I \mathbf{5}_{\phi_u}$. To achieve this we shift the charges of the first set in table I with $a = -3b = +1$, and the charges of the second set with $a = -3b = -1$. This yields the charge assignments given in table II. With these assignments, R-parity arises as an accidental symmetry enforced by $SU(5) \times U(1)_F$ gauge invariance.

	$\bar{\mathbf{5}}_1$	$\bar{\mathbf{5}}_2$	$\bar{\mathbf{5}}_3$	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	$\mathbf{5}_{\phi_u} = -\bar{\mathbf{5}}^{\phi_d}$
(1)	-4	-2	2	5/3	2/3	$-1/3$	2/3
(2)	-4	4	0	5/3	$-4/3$	$-1/3$	2/3

TABLE II: The charge assignments (1) and (2) of table I redefined to forbid the R-parity violating couplings $\bar{\mathbf{5}}_I \bar{\mathbf{5}}_J \mathbf{10}_K$ and $\bar{\mathbf{5}}_I \mathbf{5}_{\phi_u}$ by means of the shifts in eq. (12) with $a = -3b = +1$ for case (1) and $a = -3b = -1$ for case (2).

IV. THE MODEL

For the two sets of $U(1)_F$ charges in table II, charge counting suggests the following (naive) hierarchical patterns for the Yukawa matrices. For the anomalous case (1) we have

$$Y^D \sim \begin{pmatrix} \epsilon^3 & \epsilon^4 & \epsilon^5 \\ \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon \end{pmatrix}, \qquad Y^U \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad (16)$$

where in Y^D the rows correspond to $(\bar{\mathbf{5}}_1, \bar{\mathbf{5}}_2, \bar{\mathbf{5}}_3)$ and the columns to $(\mathbf{10}_1, \mathbf{10}_2, \mathbf{10}_3)$. The order of magnitude of the determinants of Y^D and Y^U is

$$\det Y^D \sim \det Y^U \sim \epsilon^6. \quad (17)$$

For the non-anomalous case (2) we have

$$Y^D \sim \begin{pmatrix} \epsilon^3 & \epsilon^6 & \epsilon^5 \\ \epsilon^5 & \epsilon^2 & \epsilon^3 \\ \epsilon^1 & \epsilon^2 & \epsilon^1 \end{pmatrix} \qquad Y^U \sim \begin{pmatrix} \epsilon^4 & \epsilon & \epsilon^2 \\ \epsilon & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad (18)$$

that gives

$$\det Y^D \sim \epsilon^6, \qquad \det Y^U \sim \epsilon^2. \quad (19)$$

In case (1) the order of magnitude of the determinants eq. (17) agrees with the phenomenological result eq. (8). In contrast, for the second set of charges, $\det Y^U$ in eq. (19) is four powers of ϵ too large. This does not necessarily imply that case (2) is not viable. As we will show, in the present framework charge counting only gives upper bounds on the parametrically suppressed couplings, and it is still possible that $Y^U_{12,21}$ in eq. (18) could be promoted to $\mathcal{O}(\epsilon^3)$ thus recovering $\det Y^U \sim \epsilon^6$. Nevertheless, in the rest of this paper we will concentrate on the anomalous case (1) that looks phenomenologically more promising, and that will allow us to illustrate all the interesting features of our scheme while dealing just with the down-quarks and leptons Yukawa matrix Y^D.

Since the form of Y^U in eq. (16) is approximately diagonal, the up-quark mass hierarchy is be accounted for by the diagonal entries, while off-diagonal entries contribute to the quark mixing matrix. Given that experimental uncertainties for the GUT scale up-quarks mass ratios are rather large, ($m_u/m_c \sim 0.002 - 0.004$, $m_c/m_t \sim 0.001 - 0.003$, see [1, 25]) we assume that these ratios can be accommodated within the model, and we concentrate on the structure of Y^D. Reconciling the matrix Y^D in eq. (16) with experimental observation appears quite challenging, firstly because of an apparent 'anomaly' in the (2,1) entry that spoils approximate diagonality (this entry is crucial since it controls the value of the Cabibbo angle) and secondly because satisfying the down-quarks and leptons mass relations within $SU(5)$ GUTs is not a trivial task. In the following we analyze the contributions of different effective operators to Y^D, showing that a phenomenologically acceptable structure, able to reproduce (approximately) the correct mass ratios and to give reasonable quark-mixings can be recovered.

A. Effective operators

We assume that a large number of vectorlike FN fields exist in various $SU(5)$ representations. Since the mass M of these fields is assumed to be larger than Λ_{GUT}, at this scale the contributions to the down-quarks and leptons mass operator $\sim \bar{\mathbf{5}}_{Ia}\, \mathbf{10}^{ab}_J\, \bar{\mathbf{5}}^{\phi_d}_b$ can be evaluated by means of insertions of pointlike propagators. We denote the contraction of two vectorlike fields in the representations $\mathbf{R}$, $\bar{\mathbf{R}}$ as

$$[\mathbf{R}^{abc\ldots}_{de\ldots}\, \bar{\mathbf{R}}^{pq\ldots}_{lmn\ldots}] = \frac{-i}{M} \mathcal{S}^{abc\ldots pq\ldots}_{de\ldots lmn\ldots}, \tag{20}$$

where all the indices are $SU(5)$ indices, and $\mathcal{S}$ is the appropriate group index structure. The structures $\mathcal{S}$ for several $SU(5)$ representations are given in appendix A. We further use $\langle \Sigma_\pm \rangle = (V/\sqrt{60}) \times \mathrm{diag}(2, 2, 2, -3, -3)$ where the factor $1/\sqrt{60}$ gives the usual normalization of the $SU(5)$ generators $\mathrm{Tr}(\mathbf{R}^a \bar{\mathbf{R}}^b) = \frac{1}{2}\delta^{ab}$. The number of effective operators rapidly increases with increasing powers of ϵ: we find 4 possible operators at $\mathcal{O}(\epsilon)$, 17 at $\mathcal{O}(\epsilon^2)$, and more than 70 at $\mathcal{O}(\epsilon^3)$ (see appendix B). To establish a path in this forest of operators, we will stick to the following rule: *At each order in ϵ, all the possible operators that can arise are allowed to contribute, unless there is a compelling reason to forbid specific contributions* [29]. To forbid one operator, we simply assume that the representations that gives rise to it does not exist.

The assumptions of a unique heavy mass parameter M and of universality of the fundamental Yukawa coupling (whose value can be absorbed in the vev V, except for an overall factor that cancels in the mass ratios) implies a high level of predictivity. Of course there are still several sources of theoretical uncertainties, and before getting into numbers let us briefly discuss the most relevant ones.

i) In general, different FN representations $\mathbf{R}$, $\mathbf{R}'$, ... can mix through terms like $\mathbf{R}\,\Sigma\,\mathbf{R}'$ thus splitting the heavy mass eigenstates. Formally this is an effect of relative order ϵ.

ii) Each entry in the Yukawa matrices receives contributions from higher order operators involving insertions of Σ_0. Assuming $\langle \Sigma_0 \rangle \sim \langle \Sigma_\pm \rangle$ also these corrections are of relative order ϵ.

iii) Universality for the fundamental Yukawa coupling holds at a scale $> \Lambda_{GUT}$. However, for different $SU(5)$ representations the renormalization group (RG) evolution down to Λ_{GUT} differ, and since rather large representations are often involved, RG effects can effectively split the GUT scale couplings.

With increasing powers of ϵ, larger group theoretical coefficients and larger FN representations are involved, and all the corrections listed above grow, and can become larger than naive estimates. Therefore, we should expect that for the lighter fermions the results will be less precise. However, note that once the field content of a model is specified, all the corrections listed above are in principle computable.

B. Bottom-tau Yukawa unification

The τ and b masses arise at $\mathcal{O}(\epsilon)$ and to a good approximation are determined by the values of Y^D_{33}. We will denote as Y^ℓ_{IJ} and Y^d_{IJ} the values of Y^D_{IJ} respectively for the leptons and quarks, since in general they differ. The tensor products relevant to identify the FN representations involved in the b-τ mass operator are

$$\bar{\mathbf{5}} \otimes \bar{\mathbf{5}}^{\phi_d} = \bar{\mathbf{25}}^{(r)} = \bar{\mathbf{10}} \oplus \bar{\mathbf{15}}, \tag{21}$$

$$\mathbf{10} \otimes \bar{\mathbf{5}}^{\phi_d} = \mathbf{50}^{(r)} = \mathbf{5} \oplus \mathbf{45}. \tag{22}$$

Some results are obtained more easily in terms of reducible representations like $\bar{\mathbf{25}}^{(r)}$ and $\mathbf{50}^{(r)}$. We use a superscript $^{(r)}$ to denote the reducible character of a representation and avoid confusion with irreducible representations of equal dimensionality. In eqs. (21) and (22) the second equalities gives the irreducible fragments. The representations conjugate to the tensor products in eqs. (21) and (22) are found respectively in the tensor products of $\mathbf{10}$ and $\bar{\mathbf{5}}$ with the $\mathbf{24}$ adjoint

$$\mathbf{10} \otimes \mathbf{24} = \mathbf{25}^{(r)} \oplus \mathbf{40} \oplus \mathbf{175}, \tag{23}$$

$$\bar{\mathbf{5}} \otimes \mathbf{24} = \bar{\mathbf{50}}^{(r)} \oplus \mathbf{70}. \tag{24}$$

Hence, at $\mathcal{O}(\epsilon)$ the contributions to the b-τ mass operator involve $\mathbf{25}^{(r)}$, $\mathbf{50}^{(r)}$ and their conjugate representations, as is diagrammatically depicted in fig. 1. In terms of irreducible representations, the following operators arise:

$$O(\epsilon; \mathbf{10}_{-4/3}) = \bar{\mathbf{5}}_a \bar{\mathbf{5}}^{\phi_d}_b \left[\mathbf{10}^{\,ab}\ \bar{\mathbf{10}}_{lm}\right] \Sigma^m_n \mathbf{10}^{nl}, \tag{25}$$

$$O(\epsilon; \mathbf{15}_{-4/3}) = \bar{\mathbf{5}}_a \bar{\mathbf{5}}^{\phi_d}_b \left[\mathbf{15}^{\,ab}\ \bar{\mathbf{15}}_{lm}\right] \Sigma^m_n \mathbf{10}^{nl}, \tag{26}$$

$$O(\epsilon;\ \mathbf{5}_{-1}) = \bar{\mathbf{5}}_a \Sigma^a_b \left[\mathbf{5}^b\ \bar{\mathbf{5}}_l\right] \bar{\mathbf{5}}^{\phi_d}_m \mathbf{10}^{lm}, \tag{27}$$

$$O(\epsilon; \mathbf{45}_{-1}) = \bar{\mathbf{5}}_a \Sigma^c_b \left[\mathbf{45}^{ba}_c\ \bar{\mathbf{45}}^n_{ml}\right] \bar{\mathbf{5}}^{\phi_d}_n \mathbf{10}^{lm}. \tag{28}$$

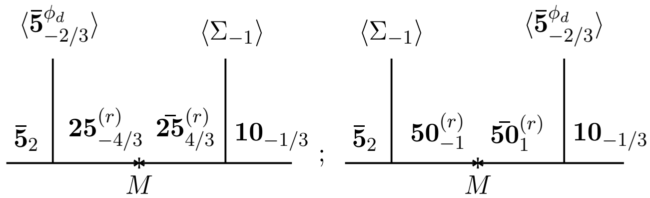

FIG. 1: Diagrammatic representation of the $\mathcal{O}(\epsilon)$ contributions to Y^D_{33}. The subscripts give the $U(1)_F$ charges.

We denote as $O(\epsilon^n;\ \mathbf{R}'_{F_1}, \mathbf{R}''_{F_2}, \ldots \mathbf{R}^n{}_{F_n})$ an $SU(5)$ invariant operator of order ϵ^n (i.e. with n-insertions of Σ) induced by a set of n pairs of FN fields in the representations $\mathbf{R}$, with charge F. The string of $\mathbf{R}$'s is ordered from left to right assuming always that the $\bar{\mathbf{5}}$ and $\mathbf{10}$ containing the SM fields are respectively the first and last element of the string. Note that together with the values of the F-charges, this determines univocally the order of insertion of $\bar{\mathbf{5}}^{\phi_d}_{-2/3}$ and $\Sigma_{\pm 1,0}$.

Using the relevant group structures $\mathcal{S}$ and the vertices $\mathcal{V}$ given in appendix A, restricting $\bar{\mathbf{5}}^{\phi_d}$ to the $SU(2)$ Higgs doublet (indices $i = 4, 5$) and projecting Σ on the vacuum $\propto \mathrm{diag}(2, 2, 2, -3, -3)$ we obtain the contributions to Y^ℓ_{33} and Y^d_{33} given in table III. It is a non-trivial result that

	$\frac{Y^\ell_{33}}{\epsilon/\sqrt{60}}$	$\frac{Y^d_{33}}{\epsilon/\sqrt{60}}$
$O(\epsilon; \mathbf{10}_{-4/3})$	6	1
$O(\epsilon; \mathbf{15}_{-4/3})$	0	5
$O(\epsilon;\ \mathbf{5}_{-1})$	-3	2
$O(\epsilon; \mathbf{45}_{-1})$	15	10
$\sum_{\mathbf{R}} O(\epsilon; \mathbf{R})$	18	18

TABLE III: Coefficients of the four operators that can contribute to Y^ℓ_{33} and Y^d_{33} at order ϵ, in units of $\epsilon/\sqrt{60}$. The equality of the two sums given in the last row corresponds to b-τ Yukawa unification.

summing up the four contributions for the leptons and for the quarks one obtains the same result. This allows us to conclude that *the most general effective operator allowed at order ϵ by the $SU(5) \times U(1)_F$ symmetry yields b-τ Yukawa unification.*

Note that unification is already preserved by the individual contributions of the two reducible representations: $O(\epsilon; \mathbf{25}^{(r)}) = O(\epsilon; \mathbf{10}) + O(\epsilon; \mathbf{15}) \to 6$ and $O(\epsilon; \mathbf{50}^{(r)}) = O(\epsilon;\ \mathbf{5}) + O(\epsilon; \mathbf{45}) \to 12$. This result can be easily verified by using the $\mathcal{S}$ structures for the reducible representations given in appendix A eqs. (A28) and (A29):

$$\bar{\mathbf{5}}_a \bar{\mathbf{5}}^{\phi_d}_b \left[\mathbf{25}^{(r)\,ab}\ \bar{\mathbf{25}}^{(r)}_{lm} \right] \Sigma^m_n \mathbf{10}^{nl} \to \frac{6\epsilon}{\sqrt{60}}\, \bar{\mathbf{5}}_a \mathbf{10}^{ai} \bar{\mathbf{5}}^{\phi_d}_i, \quad (29)$$

$$\bar{\mathbf{5}}_a \Sigma^c_b \left[\mathbf{50}^{(r)}{}^{ba}_{\ \ c}\ \bar{\mathbf{50}}^{(r)n}_{ml} \right] \bar{\mathbf{5}}^{\phi_d}_n \mathbf{10}^{lm} \to \frac{12\epsilon}{\sqrt{60}}\, \bar{\mathbf{5}}_a \mathbf{10}^{ai} \bar{\mathbf{5}}^{\phi_d}_i. \quad (30)$$

where $i = 4, 5$ corresponds to the $SU(2)$ components of $\bar{\mathbf{5}}^{\phi_d}$. We see that in both cases the coefficients are independent of the particular value of the index $a = 1, 2, 3$ (b-quark) or $a = 4$ (τ-lepton)[30]. Therefore, in terms of contributions of reducible representations, unification is preserved even if the assumption of universality of the Yukawa couplings is relaxed. Note however, that this is only a leading order result, since in general the $\mathcal{O}(\epsilon^2)$ corrections to Y^ℓ_{33} and Y^d_{33} will be different. Therefore, the models predicts that b-τ Yukawa unification is only an *approximate leading order result.*

From table III we see that for the τ-lepton the effective operator involving the $\mathbf{15}$ has a vanishing coefficient. This happens because the $\mathbf{15}$ is symmetric in $SU(5)$ indices while the $\mathbf{10}$ is antisymmetric. The $SU(2)$ singlet leptons correspond to $\mathbf{10}^{45,54}$. Specifying these indices forces $\langle \Sigma \rangle$ to the lowest 2×2 corner, that is proportional to the identity $I_{2\times 2}$, and thus because of the symmetric-antisymmetric contraction the operator vanishes. It is not difficult to prove that the same happens also for higher order diagrams containing the $\mathbf{15}$. Similar implications of $SU(5)$ index properties occur also for other representations. In particular, the effective operators involving the $\mathbf{40}$ also vanish for the leptons.

C. The Y^D_{21} entry and the Cabibbo angle

$U(1)_F$ charge counting suggests that the (2,1) entry in Y^D eq. (16) can arise already at $\mathcal{O}(\epsilon)$. This would spoil the approximate diagonality of Y^D and induce a too large mixing between the down-quarks of the first and second generations. In short, such a parametric suppression is too mild to be phenomenologically acceptable. In our scheme this entry can be generated by operators analogous to the ones in eqs. (25)-(28) but with FN fields with different charges: $O(\epsilon; \mathbf{10}_{8/3})$, $O(\epsilon; \mathbf{15}_{8/3})$, $O(\epsilon; \mathbf{5}_{+1})$ and $O(\epsilon; \mathbf{45}_{+1})$. Therefore, if such representations do not exist, at order ϵ the Y^D_{21} entry vanishes. At the next order it could be generated by means of one insertion of the neutral adjoint Σ_0. The same restrictions that forbid the $\mathcal{O}(\epsilon)$ terms also imply that only two operators are allowed: $O(\epsilon^2; \mathbf{70}_{+1}, \mathbf{15}_{+5/3})$ and $O(\epsilon^2; \mathbf{45}_{+2}, \bar{\mathbf{40}}_{+8/3})$. $\mathbf{70}_{+1}$ must exist in order to generate Y^D_{22} at $\mathcal{O}(\epsilon^2)$ (see below), therefore to forbid the first operator we assume that $\mathbf{15}_{+5/3}$ is absent. As regard the second operator, the absence of $\bar{\mathbf{40}}_{+8/3}$ would forbid it, but would also imply a strong suppression of the Y^D_{11} entry (see below). This could even be a welcome feature, since a highly suppressed (or texture zero) Y^d_{11} is one of the conditions required to obtain the GST [3] relation. However, to stick to our guiding principle of allowing for the maximum possible number of contributions (and to avoid regularities) we assume instead that $\mathbf{45}_{+2}$ is absent. With these two assumptions, Y^D_{21} vanishes also at $\mathcal{O}(\epsilon^2)$ and approximate diagonality for Y^D is recovered. In summary, by assuming that no FN fields exist with quantum numbers corresponding to $\mathbf{15}_{8/3}$, $\mathbf{10}_{8/3}$, $\mathbf{5}_{+1}$, $\mathbf{45}_{+1}$, $\mathbf{15}_{+5/3}$ and $\mathbf{45}_{+2}$, the Y^D_{21} entry is promoted to $\mathcal{O}(\epsilon^3)$. The possible oper-

ators that can contribute at this order are listed in table IV. Note that no operators involving a pair of Σ_0 is

$\cancel{\mathbf{10}}_{+8/3}$, $\cancel{\mathbf{15}}_{+8/3}$, $\cancel{\mathbf{5}}_{+1}$, $\cancel{\mathbf{45}}_{+1}$; $\cancel{\mathbf{15}}_{+5/3}$, $\cancel{\mathbf{45}}_{+2}$		
	$\frac{Y^\ell_{21}}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d_{21}}{(\epsilon/\sqrt{60})^3}$
$\left[\Sigma_{-1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{+1}\Sigma_{+1}\right]$		
$O(\epsilon^3; \mathbf{5}_{+3}, \mathbf{10}_{+11/3}, \overline{\mathbf{40}}_{+8/3})$	0	100
$O(\epsilon^3; \mathbf{45}_{+3}, \mathbf{10}_{+11/3}, \overline{\mathbf{40}}_{+8/3})$	0	500
$O^\uparrow(\epsilon^3; \mathbf{45}_{+3}, \overline{\mathbf{40}}_{+11/3}, \overline{\mathbf{40}}_{+8/3})$	0	800
$O^\downarrow(\epsilon^3; \mathbf{45}_{+3}, \overline{\mathbf{40}}_{+11/3}, \overline{\mathbf{40}}_{+8/3})$	0	-1400
$\left[\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{-1}\right]$		
$O(\epsilon^3; \mathbf{70}_{+1}, \cancel{\mathbf{5}_0}, \mathbf{10}_{+2/3})$	$\cancel{1350}$	$\cancel{200}$
$O(\epsilon^3; \mathbf{70}_{+1}, \cancel{\mathbf{5}_0}, \mathbf{15}_{+2/3})$	0	$\cancel{1000}$
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{10}_{+2/3})$	-1350	200
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \overline{\mathbf{40}}_{+2/3})$	0	4000
$O^\uparrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	-400
$O^\downarrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	800
$\left[\Sigma_{-1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{+1}\right]$		
$O(\epsilon^3; \mathbf{45}_{+3}, \cancel{\mathbf{50}_{+2}}, \overline{\mathbf{40}}_{+8/3})$	0	–
$\sum_{\mathbf{R}} O(\epsilon^3; \mathbf{R})$	-1350	4600

TABLE IV: Contributions to Y^D_{21} at $\mathcal{O}(\epsilon^3)$ in units of $(\epsilon/\sqrt{60})^3$. The upper row lists the representations in the absence of which Y^D_{21} is promoted to $\mathcal{O}(\epsilon^3)$. The three possible orders of insertion of $\bar{\mathbf{5}}^{\phi_d}$ and Σ's are specified in square brackets. Constraints from Y^D_{22} imply crossing out $\bar{\mathbf{5}}_0$, while the $\overline{\mathbf{50}}$ is crossed out because it is not included in the analysis. The sums do not include the crossed out coefficients.

allowed. Note also that when the $\mathbf{40}\Sigma\overline{\mathbf{40}}$ and $\overline{\mathbf{70}}\Sigma\mathbf{70}$ vertices are involved, two different contributions are present (respectively, the third and fourth, ninth and tenth entries in table IV). This is because for these vertices two inequivalent contractions of the $SU(5)$ indices are possible (see eqns. (A7)-(A9) in appendix A). This is always the case when a representation is contained twice in its tensor product with the adjoint $\mathbf{R}\otimes\Sigma = \mathbf{R}\oplus\mathbf{R}\oplus\dots$ ($\mathbf{R} = \overline{\mathbf{40}}$, $\mathbf{45}$, $\mathbf{70}\dots$). We distinguish the two contributions by means of up- and down-arrow labels $O^\uparrow$, $O^\downarrow$.

D. The Y^D_{22} entry and the strange and muon masses

After the FN representations are restricted according with the previous discussion, at $\mathcal{O}(\epsilon^2)$ only two contributions to Y^D_{22} remain possible. They are given in table V. We see that for the leptons a cancellation occurs.

	$\frac{Y^\ell_{22}}{(\epsilon/\sqrt{60})^2}$	$\frac{Y^d_{22}}{(\epsilon/\sqrt{60})^2}$
$\left[\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O(\epsilon^2; \mathbf{70}_{+1}, \cancel{\mathbf{5}_0})$	$-\cancel{225}$	$-\cancel{200}$
$O(\epsilon^2; \mathbf{70}_{+1}, \mathbf{45}_0)$	225	-200

TABLE V: Contributions to Y^D_{22} at $\mathcal{O}(\epsilon^2)$ in units of $(\epsilon/\sqrt{60})^2$. According to the discussion in the text, the operator involving the $\mathbf{5}_0$ has been crossed out.

(This cancellation can also be traced back to the symmetry/antisymmetry in the two upper indices respectively of the $\mathbf{70}$ and of the reducible $\mathbf{50}^{(r)} = \mathbf{5}\oplus\mathbf{45}$.) If both contributions are allowed, m_μ would be formally suppressed to $\mathcal{O}(\epsilon^3)$. Therefore we assume that $\mathbf{5}_0$ is absent, and we keep only the contribution of the $\mathbf{45}_0$. Having estimated Y^ℓ_{33} and Y^ℓ_{22}, we can now fit ϵ to the value of the μ and τ mass ratio $Y^\ell_{33}/Y^\ell_{22} \sim m_\mu/m_\tau \sim 0.06$ obtaining

$$\epsilon \sim \frac{18\sqrt{60}}{225}\frac{m_\mu}{m_\tau} \simeq 0.037. \tag{31}$$

The ratio between the top and bottom Yukawa couplings can be estimated using this value of ϵ:

$$\frac{Y^U_{33}}{Y^d_{33}} \sim \left(\frac{18\epsilon}{\sqrt{60}}\right)^{-1} \simeq 12. \tag{32}$$

Note that this result points towards moderate values of $\tan\beta$ ($\tan\beta < 10$ [1]). From the sum for the contributions in table IV and from table V we can also see that the down-quark sector gives a sizeable contribution to Cabibbo mixing $Y^d_{21}/Y^d_{22} \sim 0.11$. As regards the mass

	$\frac{Y^\ell_{22}}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d_{22}}{(\epsilon/\sqrt{60})^3}$
$\left[\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_0\right]$		
$O^\uparrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	-400
$O^\downarrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	800
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{10}_{+2/3})$	-1350	200
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \overline{\mathbf{40}}_{+2/3})$	0	4000
$\left[\Sigma_0\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O(\epsilon^3; \mathbf{5}_{+2}, \mathbf{70}_{+1}, \mathbf{45}_0)$	675	400
$O^\uparrow(\epsilon^3; \mathbf{70}_{+2}, \mathbf{70}_{+1}, \mathbf{45}_0)$	4725	800
$O^\downarrow(\epsilon^3; \mathbf{70}_{+2}, \mathbf{70}_{+1}, \mathbf{45}_0)$	675	-1600
$\left[\Sigma_{+1}\Sigma_0\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O^\uparrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_{+1}, \mathbf{45}_0)$	4725	800
$O^\downarrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_{+1}, \mathbf{45}_0)$	675	-1600
$\left[\Sigma_{+1}\Sigma_{+1}\Sigma_0\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O^\uparrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{45}_0)$	4275	1200
$O^\downarrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{45}_0)$	1575	-400
$O^\uparrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{45}_0)$	4275	800
$O^\downarrow(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{45}_0)$	675	-1600
$\sum_{\mathbf{R}} O(\epsilon^3; \mathbf{R})$	20925	3400

TABLE VI: Possible $\mathcal{O}(\epsilon^3)$ corrections to Y^D_{22}.

ratio m_s/m_μ we see from table V that a rather large value is obtained. However, by inspecting the set of $\mathcal{O}(\epsilon^3)$ corrections we have found that they can add up to produce a surprisingly large coefficient $20925/(\sqrt{60})^3 \sim 45$ (see table VI). This can increase m_μ by 50% and bring the value of $3m_s/m_\mu$ quite close to the range given in eq. (2). This gives one example of a case when a non-hierarchical coefficient, rather than being an $\mathcal{O}(1)$ number, is large enough to compensate for one additional factor of ϵ.

E. The Y^D_{11} entry and the down and electron masses

The contributions to the down-quark and electron masses are listed in table VII. We see that all the four possibilities involve a $\bar{\mathbf{4}}\mathbf{0}$ and, as was mentioned above, because of the $SU(5)$ indices properties of the $\mathbf{40}$ at this order the electron mass vanishes. Y^ℓ_{11} is thus promoted to $\mathcal{O}(\epsilon^4)$ and this also implies $\det Y^\ell \sim \epsilon^7$ instead than the naive estimate eq. (17). It is interesting that in the present scheme the FN mechanism is not only able to split the $SU(5)$ mass degeneracies by means of non hierarchical coefficients, but it can also produce a relative hierarchy between the lepton and down-type quark masses of the same generation. As regards the second GUT relation in eq. (2), it can be reproduced if, for example, the $\mathcal{O}(\epsilon^4)$ correction to the down-quark Yukawa coupling remains small, while the coefficient for the electron is ≈ 15, and we have just seen that numbers of this size are certainly possible.

	$\frac{Y^\ell_{11}}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d_{11}}{(\epsilon/\sqrt{60})^3}$
$\left[\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{+1}\Sigma_{+1}\right]$		
$O(\epsilon^3;\mathbf{5}_{+3},\mathbf{10}_{+11/3},\bar{\mathbf{4}}\mathbf{0}_{+8/3})$	0	100
$O(\epsilon^3;\mathbf{45}_{+3},\mathbf{10}_{+11/3},\bar{\mathbf{4}}\mathbf{0}_{+8/3})$	0	500
$\left[\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{+1}\Sigma_{+1}\Sigma_{+1}\right]$		
$O(\epsilon^3;\mathbf{10}_{+14/3},\mathbf{10}_{+11/3},\bar{\mathbf{4}}\mathbf{0}_{+8/3})$	0	50
$O(\epsilon^3;\mathbf{15}_{+14/3},\mathbf{10}_{+11/3},\bar{\mathbf{4}}\mathbf{0}_{+8/3})$	0	250
$\sum_{\mathbf{R}} O(\epsilon^3;\mathbf{R})$	0	900

TABLE VII: Operators contributing to Y^D_{11} at $\mathcal{O}(\epsilon^3)$.

F. Other $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ entries: Y^D_{32}, Y^D_{31} and Y^D_{23}.

For completeness, we list in tables VIII, IX and X the coefficients of the operators contributing respectively to Y^D_{32} at $\mathcal{O}(\epsilon^2)$ and to Y^D_{31} and Y^D_{23} at $\mathcal{O}(\epsilon^3)$. The last two entries Y^D_{12} and Y^D_{13} are highly suppressed (at least as ϵ^4 and ϵ^5) and we have not computed them. In any case they give only negligible corrections to mass ratios and mixing angles.

V. DISCUSSION

Before discussing what can be learned from our results, let us resume briefly the main steps of the whole procedure. We have selected a set of $U(1)_F$ charges suitable to reproduce the observed fermion mass hierarchy eqs. (6) and (7), and satisfying our theoretical prejudice that each fermion multiplet should be univocally identified by the GUT-flavor symmetry. By assuming a common mass for the heavy states and universality for the fundamental Yukawa couplings we have reduced the number of free parameters to one: the dimensionless symmetry breaking parameter ϵ. We have then computed the effective down-quarks and lepton Yukawa matrices by including at each

	$\frac{Y^\ell_{32}}{(\epsilon/\sqrt{60})^2}$	$\frac{Y^d_{32}}{(\epsilon/\sqrt{60})^2}$
$\left[\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{-1}\Sigma_{-1}\right]$		
$O(\epsilon^2;\mathbf{10}_{-4/3},\mathbf{10}_{-1/3})$	−36	−1
$O(\epsilon^2;\mathbf{10}_{-4/3},\mathbf{15}_{-1/3})$	0	−25
$O(\epsilon^2;\mathbf{10}_{-4/3},\bar{\mathbf{4}}\mathbf{0}_{-1/3})$	0	−50
$O(\epsilon^2;\mathbf{15}_{-4/3},\mathbf{10}_{-1/3})$	0	−5
$O(\epsilon^2;\mathbf{15}_{-4/3},\mathbf{15}_{-1/3})$	0	5
$\left[\Sigma_{-1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{-1}\right]$		
$O(\epsilon^2;\mathbf{5}_{-1},\mathbf{10}_{-1/3})$	18	−2
$O(\epsilon^2;\mathbf{5}_{-1},\mathbf{15}_{-1/3})$	0	−10
$O(\epsilon^2;\mathbf{45}_{-1},\mathbf{10}_{-1/3})$	−90	−10
$O(\epsilon^2;\mathbf{45}_{-1},\bar{\mathbf{4}}\mathbf{0}_{-1/3})$	0	−200
$O(\epsilon^2;\mathbf{70}_{-1},\mathbf{15}_{-1/3})$	0	100
$\left[\Sigma_{-1}\Sigma_{-1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O(\epsilon^2;\mathbf{5}_{-1},\mathbf{45}_{0})$	45	−20
$O^\uparrow(\epsilon^2;\mathbf{45}_{-1},\mathbf{45}_{0})$	285	−60
$O^\downarrow(\epsilon^2;\mathbf{45}_{-1},\mathbf{45}_{0})$	105	20
$O(\epsilon^2;\mathbf{70}_{-1},\mathbf{45}_{0})$	225	−200
$\sum_{\mathbf{R}} O(\epsilon^2;\mathbf{R})$	552	−458

TABLE VIII: Operators contributing to Y^D_{32} at $\mathcal{O}(\epsilon^2)$.

	$\frac{Y^\ell_{31}}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d_{31}}{(\epsilon/\sqrt{60})^3}$
$\left[\Sigma_{-1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{-1}\Sigma_{-1}\right]$		
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{10}_{-1/3},\mathbf{10}_{+2/3})$	−108	2
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{10}_{-1/3},\mathbf{15}_{+2/3})$	0	50
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{10}_{-1/3},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	100
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{15}_{-1/3},\mathbf{10}_{+2/3})$	0	10
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{15}_{-1/3},\mathbf{15}_{+2/3})$	0	−10
$O(\epsilon^3;\mathbf{45}_{-1},\mathbf{10}_{-1/3},\mathbf{10}_{+2/3})$	540	10
$O(\epsilon^3;\mathbf{45}_{-1},\mathbf{10}_{-1/3},\mathbf{15}_{+2/3})$	0	250
$O(\epsilon^3;\mathbf{45}_{-1},\mathbf{10}_{-1/3},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	500
$O(\epsilon^3;\mathbf{45}_{-1},\bar{\mathbf{4}}\mathbf{0}_{-1/3},\mathbf{10}_{+2/3})$	0	−200
$O^\uparrow(\epsilon^3;\mathbf{45}_{-1},\bar{\mathbf{4}}\mathbf{0}_{-1/3},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	800
$O^\downarrow(\epsilon^3;\mathbf{45}_{-1},\bar{\mathbf{4}}\mathbf{0}_{-1/3},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	−1400
$O(\epsilon^3;\mathbf{70}_{-1},\mathbf{15}_{-1/3},\mathbf{10}_{+2/3})$	0	−100
$O(\epsilon^3;\mathbf{70}_{-1},\mathbf{15}_{-1/3},\mathbf{15}_{+2/3})$	0	100
$\left[\Sigma_{-1}\Sigma_{-1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{-1}\right]$		
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{45}_{0},\mathbf{10}_{+2/3})$	−270	20
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{45}_{0},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	400
$O(\epsilon^3;\mathbf{5}_{-1},\mathbf{70}_{0},\mathbf{15}_{+2/3})$	0	−200
$O^\uparrow(\epsilon^3;\mathbf{45}_{-1},\mathbf{45}_{0},\mathbf{10}_{+2/3})$	−1710	60
$O^\downarrow(\epsilon^3;\mathbf{45}_{-1},\mathbf{45}_{0},\mathbf{10}_{+2/3})$	−630	−20
$O^\uparrow(\epsilon^3;\mathbf{45}_{-1},\mathbf{45}_{0},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	−1200
$O^\downarrow(\epsilon^3;\mathbf{45}_{-1},\mathbf{45}_{0},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	400
$O(\epsilon^3;\mathbf{45}_{-1},\cancel{\bar{\mathbf{5}}\mathbf{0}_{0}},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	–	–
$O(\epsilon^3;\mathbf{45}_{-1},\mathbf{70}_{0},\mathbf{15}_{+2/3})$	0	1000
$O(\epsilon^3;\mathbf{70}_{-1},\mathbf{45}_{0},\mathbf{10}_{+2/3})$	−1350	200
$O(\epsilon^3;\mathbf{70}_{-1},\mathbf{45}_{0},\bar{\mathbf{4}}\mathbf{0}_{+2/3})$	0	4000
$O^\uparrow(\epsilon^3;\mathbf{70}_{-1},\mathbf{70}_{0},\mathbf{15}_{+2/3})$	0	−400
$O^\downarrow(\epsilon^3;\mathbf{70}_{-1},\mathbf{70}_{0},\mathbf{15}_{+2/3})$	0	800
$\sum_{\mathbf{R}} O(\epsilon^3;\mathbf{R})$	−3528	5172

TABLE IX: Operators contributing to Y^D_{31} at $\mathcal{O}(\epsilon^3)$.

	$\frac{Y^{\ell}_{23}}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^{d}_{23}}{(\epsilon/\sqrt{60})^3}$
$\left[\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\Sigma_{+1}\right]$		
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{10}_{+2/3})$	-1350	200
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \bar{\mathbf{40}}_{+2/3})$	0	4000
$O^{\uparrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	-400
$O^{\downarrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{15}_{+2/3})$	0	800
$\left[\Sigma_{+1}\Sigma_{+1}\Sigma_{+1}\bar{\mathbf{5}}^{\phi_d}_{-2/3}\right]$		
$O(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{5}_{-1})$	225	-200
$O^{\uparrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{45}_{-1})$	4275	1200
$O^{\downarrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{45}_0, \mathbf{45}_{-1})$	1575	-400
$O^{\uparrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{5}_{-1})$	-4725	800
$O^{\downarrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{5}_{-1})$	-675	-1600
$O^{\uparrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{45}_{-1})$	4725	800
$O^{\downarrow}(\epsilon^3; \mathbf{70}_{+1}, \mathbf{70}_0, \mathbf{45}_{-1})$	675	-1600
$\sum_{\mathbf{R}} O(\epsilon^3; \mathbf{R})$	4724	3600

TABLE X: Operators contributing to Y^D_{23} at $\mathcal{O}(\epsilon^3)$.

order in ϵ all the possible operators, except for a few cases when eliminating some contribution was mandatory (this was done consistently, by assuming that FN fields in specific $SU(5) \times U(1)_F$ representations are absent). We have seen that at leading order b-τ unification is preserved, while at order ϵ^2 and higher the lepton and down-quark Yukawa matrices differ, and not only in the non-hierarchical coefficients, but possibly also in the order of their hierarchical suppression.

The lepton Yukawa matrix Y^ℓ that we have obtained is not particularly predictive. This is because the ratio m_μ/m_τ has been fitted to determine the value of ϵ, m_e got promoted to $\mathcal{O}(\epsilon^4)$ and, since we have limited our analysis to $\mathcal{O}(\epsilon^3)$, has not been computed, and quantitative results for the leptonic mixing angles require including a model for neutrino masses. This implies more structure and additional assumptions, and goes beyond the scope of this study. The down-quarks Yukawa matrix Y^d is more informative. Numerically we obtain

$$Y^d \approx \begin{pmatrix} 1.9\,\epsilon^3 & \sim \epsilon^5 & \sim \epsilon^4 \\ 9.9\,\epsilon^3 & -3.3\,\epsilon^2 & 7.8\,\epsilon^3 \\ 11.1\,\epsilon^3 & -7.6\,\epsilon^2 & 2.3\,\epsilon \end{pmatrix}, \qquad (33)$$

where $\epsilon \sim 0.037$. From Y^d we obtain the mass ratios

$$\frac{m_s}{m_b} \approx 0.05, \qquad \frac{m_d}{m_s} \approx 0.02, \qquad (34)$$

together with the down-quarks L-handed mixing matrix

$$V^d_L \approx \begin{pmatrix} 0.99 & 0.11 & 0.007 \\ 0.11 & -0.98 & -0.12 \\ 0.006 & -0.12 & 0.99 \end{pmatrix}. \qquad (35)$$

The ratios in eq. (34) suggest that m_s is about a factor of 2 too large (experimentally $m_s/m_b \sim 0.01$-0.02, $m_d/m_s \sim 0.04$-0.06). This is also suggested by the GUT relation $3m_s/m_\mu$ whose central value in eq. (2) would be reproduced rather precisely if m_s were half its size. As regards the mixing matrix V^d_L, it has a quite reasonable structure: if the corresponding matrix in the up-quark sector has a similar structure, it is likely that the CKM matrix could be correctly reproduced. Of course, one could improve the numerical performance of the model by inspecting carefully tables III to X and eliminating (consistently) specific contributions. However, in our opinion there is not much to learn from the construction of an *ad hoc* realization, even if quantitatively successful. For example, it would not be surprising if starting from the second set of charges in table II, and with a careful choice of the relevant contributions, one could also obtain acceptable results. Also, other charge assignments different from the ones given in table II could be viable since, as we have learned, starting from a set of charges that yields a (naive) hierarchy milder than the one observed, it can still be possible to generate the correct hierarchical pattern eqs. (6) and (7).

Instead, we think that something more interesting can be learned by considering some general features of the model. The Abelian flavor symmetry was introduced to generate a hierarchy between the entries of the Yukawa matrices. While it is generally believed that from the observed hierarchy it should be possible to reconstruct the Abelian charges, we have shown that in some cases there is no direct relation between the charges and the hierarchical suppression. As regards the non-hierarchical coefficients, they are ultimately determined by the $SU(5)$ symmetry. However, a glance at Y^d in eq. (33) shows that we should not expect to observe any clear trace of this symmetry in experimentally measurable quantities. This is because the number of $SU(5)$ coefficients contributing to Y^U, Y^d and Y^ℓ is much larger than the number of entries, and in turn the number of entries is much larger than the number of observables. It is then conceivable that the unsuccess in trying to understand the origin of fermion masses could be due to the very nature of a problem in which the amount of physically accessible information is not sufficient to identify the solution. In our example, in spite of the fact that there is only one free parameter and that everything else is computable, identifying the simple $SU(5) \times U(1)_F$ symmetry could well remain out of the reach of theoretical efforts. However, if in the future more precise measurements will confirm with high precision some of the observed regularities, and if new regularities will emerge, this would be a convincing hint that only a few fundamental parameters concur to determine the fermion mass spectrum, and would disprove schemes like the one we have discussed.

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APPENDIX A: GROUP THEORY

1. Tensor products

We list some useful tensor products involving the $\bar{\mathbf{5}}^{\phi_d}$ and the **24**-dimensional adjoint Σ containing the Higgs fields (in some cases our conventions for the conjugate representations differ from the ones used in [24].)

$$\begin{aligned}
\overline{\mathbf{5}} \otimes \overline{\mathbf{5}} &= \overline{\mathbf{10}} \oplus \overline{\mathbf{15}} \\
\mathbf{10} \otimes \overline{\mathbf{5}} &= \mathbf{5} \oplus \mathbf{45} \\
\overline{\mathbf{10}} \otimes \overline{\mathbf{5}} &= \mathbf{10} \oplus \overline{\mathbf{40}} \\
\mathbf{15} \otimes \overline{\mathbf{5}} &= \mathbf{5} \oplus \mathbf{70} \\
\overline{\mathbf{15}} \otimes \overline{\mathbf{5}} &= \overline{\mathbf{35}} \oplus \overline{\mathbf{40}} \\
\mathbf{40} \otimes \overline{\mathbf{5}} &= \mathbf{10} \oplus \mathbf{15} \oplus \mathbf{175} \\
\overline{\mathbf{40}} \otimes \overline{\mathbf{5}} &= \mathbf{45} \oplus \overline{\mathbf{50}} \oplus \overline{\mathbf{105}} \\
\overline{\mathbf{45}} \otimes \overline{\mathbf{5}} &= \overline{\mathbf{10}} \oplus \mathbf{40} \oplus \overline{\mathbf{175}} \\
\overline{\mathbf{70}} \otimes \overline{\mathbf{5}} &= \overline{\mathbf{15}} \oplus \overline{\mathbf{160}} \oplus \overline{\mathbf{175}} \\
\mathbf{5} \otimes \mathbf{24} &= \mathbf{5} \oplus \mathbf{45} \oplus \mathbf{70} \\
\mathbf{10} \otimes \mathbf{24} &= \mathbf{10} \oplus \mathbf{15} \oplus \overline{\mathbf{40}} \oplus \mathbf{175} \\
\mathbf{15} \otimes \mathbf{24} &= \mathbf{10} \oplus \mathbf{15} \oplus \mathbf{160} \oplus \mathbf{175} \\
\mathbf{40} \otimes \mathbf{24} &= \overline{\mathbf{10}} \oplus \mathbf{35} \oplus \mathbf{40} \oplus \mathbf{40} \oplus \overline{\mathbf{175}} \oplus \overline{\mathbf{210}} \oplus \mathbf{450}' \\
\mathbf{45} \otimes \mathbf{24} &= \mathbf{5} \oplus \mathbf{45} \oplus \mathbf{45} \oplus \overline{\mathbf{50}} \oplus \mathbf{70} \oplus \overline{\mathbf{105}} \oplus \mathbf{280} \oplus \overline{\mathbf{480}} \\
\mathbf{70} \otimes \mathbf{24} &= \mathbf{5} \oplus \mathbf{45} \oplus \mathbf{70} \oplus \mathbf{70} \oplus \mathbf{280} \oplus \mathbf{280}' \oplus \overline{\mathbf{450}} \oplus \overline{\mathbf{480}}.
\end{aligned} \tag{A1}$$

2. Vertices

The fundamental vertices involve $\bar{\mathbf{5}}_a^{\phi_d}$ and the adjoint Σ_b^a. They have the general form $-i\lambda\mathcal{V}$ where λ is assumed universal and $\mathcal{V} = \mathbf{R}\bar{\mathbf{5}}^{\phi_d}\mathbf{R}'$ or $\mathbf{R}\Sigma\mathbf{R}'$, with $\mathbf{R}, \mathbf{R}' = \mathbf{5}, \mathbf{10}, \mathbf{15}, \mathbf{45}, \ldots$. The relevant field contractions $\mathcal{V}$ including their symmetry factors are:

$$\bar{\mathbf{5}}_a^{\phi_d}\bar{\mathbf{5}}_b\mathbf{10}^{ba} \qquad \bar{\mathbf{5}}_a^{\phi_d}\bar{\mathbf{5}}_b\mathbf{15}^{ba} \tag{A2}$$

$$\frac{1}{2}\,\bar{\mathbf{5}}_a^{\phi_d}\bar{\mathbf{45}}_{bc}^a\mathbf{10}^{cb} \qquad \frac{1}{2}\,\bar{\mathbf{5}}_a^{\phi_d}\bar{\mathbf{70}}_{bc}^a\mathbf{15}^{cb} \tag{A3}$$

$$\frac{1}{2}\bar{\mathbf{5}}_a\bar{\mathbf{45}}_{bc}^n\bar{\mathbf{40}}_{nqr}\epsilon^{abcqr} = -\frac{1}{4}\bar{\mathbf{5}}_a\bar{\mathbf{45}}_{bc}^n\bar{\mathbf{40}}_{qrn}\epsilon^{abcqr} \tag{A4}$$

$$\bar{\mathbf{5}}_a\Sigma_b^a\mathbf{5}^b \qquad \bar{\mathbf{5}}_a\Sigma_b^c\mathbf{45}_c^{ba} \qquad \bar{\mathbf{5}}_a\Sigma_b^c\mathbf{70}_c^{ba} \tag{A5}$$

$$\bar{\mathbf{10}}_{ab}\Sigma_c^b\mathbf{10}^{ca} \qquad \mathbf{15}_{ab}\Sigma_c^b\mathbf{15}^{ca} \qquad \bar{\mathbf{15}}_{ab}\Sigma_c^b\mathbf{10}^{ca} \tag{A6}$$

$$\mathbf{40}^{abc}\Sigma^{\uparrow}{}_c^d\bar{\mathbf{40}}_{dba} \qquad \mathbf{40}^{abc}\Sigma^{\downarrow}{}_a^d\bar{\mathbf{40}}_{dbc} \tag{A7}$$

$$\bar{\mathbf{45}}_{ab}^c\Sigma^{\uparrow}{}_d^b\mathbf{45}_c^{da} \qquad \frac{1}{2}\,\bar{\mathbf{45}}_{ab}^c\Sigma^{\downarrow}{}_c^d\mathbf{45}_d^{ba} \tag{A8}$$

$$\bar{\mathbf{70}}_{ab}^c\Sigma^{\uparrow}{}_d^b\mathbf{70}_c^{da} \qquad \frac{1}{2}\,\bar{\mathbf{70}}_{ab}^c\Sigma^{\downarrow}{}_c^d\mathbf{70}_d^{ba} \tag{A9}$$

$$\frac{1}{2}\mathbf{40}^{abc}\Sigma_b^d\mathbf{10}^{fg}\epsilon_{acdfg} = \frac{1}{4}\mathbf{40}^{abc}\Sigma_c^d\mathbf{10}^{fg}\epsilon_{abdfg} \tag{A10}$$

$$\bar{\mathbf{45}}_{ab}^c\Sigma_d^b\mathbf{70}_c^{da}. \tag{A11}$$

There are two inequivalent way of contracting the indices for the vertices involving the Σ with pairs of **40**, **45** and **70**. They are distinguished in eqs. (A7),(A8) and (A9) by an up- ($\Sigma^{\uparrow}$) or down-arrow ($\Sigma^{\downarrow}$) label. This can be traced back to the fact that these representations are contained twice in their tensor products with the adjoint (see the last three lines in (A1)). At order higher than ϵ^3 other representations and other vertices can appear, like e.g.

$$\frac{1}{2}\,\mathbf{35}^{abc}\Sigma_c^d\bar{\mathbf{35}}_{dab}, \quad \mathbf{40}^{abc}\Sigma_b^d\bar{\mathbf{35}}_{dac}, \quad \text{etc}\ldots \tag{A12}$$

3. Pointlike propagators

The pointlike propagators in momentum space needed to build the effective operators are defined as $(-i/M)\,\mathcal{S}_{lmn\ldots}^{abc\ldots}$ where $\mathcal{S}$ denotes the index structure appropriate for the given FN representations. The $\mathcal{S}$-factors for FN fields in the **5** and **10** can be derived with standard path integral methods, and are given in eqs. (A21) and (A23). The **15** is obtained by symmetrizing the **10** over $SU(5)$ indices, yielding eq. (A24). The sum of **10** and **15** corresponds to the reducible $\mathbf{25}^{(r)}$ and is given in eq. (A28). The tensor product $\mathbf{10}^{ab}\otimes\bar{\mathbf{5}}_c = [\mathbf{45}\oplus\mathbf{5}]_c^{ab}$ corresponds to the reducible ${\mathbf{50}^{(r)}}_c^{ab}$ antisymmetric in the two upper indices given in eq. (A29). The irreducible fragment $\mathbf{45}_c^{ab}$ can be identified by singling out the 'trace' part ${\mathbf{50}^{(r)}}_a^{ab}$ that corresponds to the irreducible $\mathbf{5}^b$ fragment. Requiring $\mathbf{45}_a^{ab} = \mathbf{45}_b^{ab} = 0$ we obtain:

$$\begin{aligned}
[\mathbf{45}_c^{ab}\,\bar{\mathbf{45}}_{lm}^n] \to -4\,\delta_c^n\,\left[\delta_l^a\,\delta_m^b - \delta_m^a\,\delta_l^b\right]_{\mathbf{50}^{(r)}} - \\
\left[\delta_c^a\,(\delta_l^b\,\delta_m^n - \delta_m^b\,\delta_l^n) - \delta_c^b\,(\delta_l^a\,\delta_m^n - \delta_m^a\,\delta_l^n)\right]_{\mathbf{5}}.
\end{aligned} \tag{A13}$$

The first term on the r.h.s corresponds to the $\mathbf{50}^{(r)}$ and the second term is the **5** piece. The overall normalization is fixed by requiring that the **5** piece gives the same contribution to the operator $\bar{\mathbf{5}}\,\mathbf{10}\,\bar{\mathbf{5}}^{\phi_d}$ than eq. (A21). By means of the identity

$$\begin{aligned}
\frac{1}{2!}\epsilon^{abnij}\,\epsilon_{clmij} = \delta_c^a\,(\delta_l^b\,\delta_m^n - \delta_m^b\,\delta_l^n) \\
-\delta_c^b\,(\delta_l^a\,\delta_m^n - \delta_m^a\,\delta_l^n) + \delta_c^n\,(\delta_l^a\,\delta_m^b - \delta_m^a\,\delta_l^b),
\end{aligned} \tag{A14}$$

(A13) can be conveniently rewritten as in eq. (A22). The **70** is constructed in a similar way. It is contained in the tensor product $\mathbf{15}^{ab}\otimes\bar{\mathbf{5}}_c = [\mathbf{70}\oplus\mathbf{5}]_c^{ab}$ that corresponds to the reducible ${\mathbf{75}^{(r)}}_c^{ab}$ symmetric in the two upper indices. By imposing the 'traceless' condition $\mathbf{70}_a^{ab} = \mathbf{70}_b^{ab} = 0$ we obtain

$$\begin{aligned}
[\mathbf{70}_c^{ab}\,\bar{\mathbf{70}}_{lm}^n] \to 6\,\delta_c^n\,\left[\delta_l^a\,\delta_m^b + \delta_m^a\,\delta_l^b\right]_{\mathbf{75}^{(r)}} - \\
\left[\delta_c^a\,(\delta_l^b\,\delta_m^n + \delta_m^b\,\delta_l^n) + \delta_c^b\,(\delta_l^a\,\delta_m^n + \delta_m^a\,\delta_l^n)\right]_{\mathbf{5}}.
\end{aligned} \tag{A15}$$

To fix the normalization one has to go to $\mathcal{O}(\epsilon^2)$ and compute e.g. the entry (11) in table B.

The **40** is contained in the tensor product $\mathbf{10}^{ab}\otimes\mathbf{5}^c = [\bar{\mathbf{10}}\oplus\mathbf{40}]^{abc}$ that corresponds to a reducible ${\mathbf{50}'^{(r)}}^{abc}$ antisymmetric in the first two indices. The three-index $\bar{\mathbf{10}}^{abc}$ fragment that we need to subtract in order to single

out the **40** is related to the two-index $\bar{\mathbf{10}}_{ij}$ through the conjugate of the following dual relations:

$$\mathbf{10}_{abc} = \frac{1}{2!}\epsilon_{abcij}\mathbf{10}^{ij}, \qquad \mathbf{10}^{ij} = \frac{1}{3!}\epsilon^{ijabc}\mathbf{10}_{abc}. \quad \text{(A16)}$$

The dual representations satisfy the identity

$$\frac{1}{3!}\mathbf{10}_{lmn}\bar{\mathbf{10}}^{abc}\epsilon_{abcij} = \frac{1}{2!}\epsilon_{lmndf}\mathbf{10}^{df}\bar{\mathbf{10}}_{ij} = \epsilon_{lmnij} \quad \text{(A17)}$$

where in the last step eq. (A23) has been used. This implies

$$\left[\mathbf{10}_{lmn}\,\bar{\mathbf{10}}^{abc}\right] \to \frac{1}{2!}\,\epsilon_{pqlmn}\epsilon^{pqabc}, \qquad \text{(A18)}$$

as can be easily checked by substituting this result in (A17) and by using $\epsilon^{pqabc}\epsilon_{abcij} = 3!\,(\delta^p_i\delta^q_j - \delta^p_j\delta^q_i)$. The expression for the **40** can be now obtained by subtracting the contribution of the $\bar{\mathbf{10}}$ from the $\mathbf{50'}^{(r)}$:

$$[\mathbf{40}^{abc}\bar{\mathbf{40}}_{lmn}] \to 3(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l)\delta^c_n - \frac{1}{2}\epsilon^{ijabc}\epsilon_{ijlmn}. \quad \text{(A19)}$$

This expression satisfies antisymmetry in the first two indices $\mathbf{40}^{abc} = -\mathbf{40}^{bac}$ plus the 10 conditions $\epsilon_{ijabc}\mathbf{40}^{abc} = 0$ that can be used to fix the factor of 3 for the $\mathbf{50'}^{(r)}$. Note that the last 10 conditions imply that the **40** does not contribute to the lepton mass operators. This can be understood by considering the vertex $\mathbf{40}^{abc}\,\Sigma^d_b\,\mathbf{10}^{fg}\epsilon_{acdfg}$ eq. (A8). When the $\mathbf{10}^{fg}$ is projected on the leptons $(f,g=4,5)$, $\langle\Sigma\rangle$ gets restricted to the upper-left 3×3 corner $(b,d=1,2,3)$ that is proportional to the identity δ^d_b. Then the vertex collapses into the 10 vanishing conditions.

The irreducible **50** (with Young tableau ⊞) is a four index representation that appears at $\mathcal{O}(\epsilon^3)$ but only in one case (the entry (38) in table XIV). Constructing its index structure and normalization is rather awkward so we have omitted the **50** from our analysis.

At $\mathcal{O}(\epsilon^4)$ the **35** can appear. Even if our analysis is restricted to $\mathcal{O}(\epsilon^3)$, we present the $\mathcal{S}$ structure for the **35** since it can be derived rather easily. The **35** is contained in the tensor product $\mathbf{15}^{ab}\otimes\mathbf{5}^b = [\mathbf{35}\oplus\mathbf{40}]^{abc}$ and corresponds to the Young tableau ☐☐☐. Being totally symmetric, its index structure is straightforwardly constructed (see eq. (A25)). The overall normalization is fixed by requiring that the **40** irreducible fragment contained in the two-index symmetric reducible $\mathbf{75'}^{(r)}$ $[\mathbf{75'}^{(r)\,abc}\,\bar{\mathbf{75}}'^{(r)}_{lmn}] \sim \left(\delta^a_l\,\delta^b_m + \delta^a_m\,\delta^b_l\right)\delta^c_n$ reproduces the results obtained with eq. (A26). The **40** in the two-index symmetric representation can be singled out by imposing the vanishing of the symmetric combinations $\mathbf{40}^{abc}+\mathbf{40}^{acb}+\mathbf{40}^{cba}=0$. Taking into account the symmetry in the first two indices this can be expressed as a cyclical relation and gives 35 conditions. We obtain

$$\begin{aligned}[\mathbf{40}^{abc}\,\bar{\mathbf{40}}_{lmn}] \to\ & 4\left[\left(\delta^a_l\,\delta^b_m + \delta^a_m\,\delta^b_l\right)\delta^c_n\right]_{\mathbf{75'}^{(r)}} - \\ & 2\left[\left(\delta^a_l\,\delta^c_m + \delta^a_m\,\delta^c_l\right)\delta^b_n + \left(\delta^c_l\,\delta^b_m + \delta^c_m\,\delta^b_l\right)\delta^a_n\right]_{\mathbf{35}}. \quad \text{(A20)}\end{aligned}$$

In summary, the relevant index structures that we have evaluated are:

$$[\mathbf{5}^a\,\bar{\mathbf{5}}_b] \to \delta^a_b \qquad \text{(A21)}$$

$$[\mathbf{45}^{ab}_c\,\bar{\mathbf{45}}^n_{lm}] \to -3\left(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l\right)\delta^n_c - \frac{1}{2}\epsilon^{abnij}\epsilon_{lmcij} \quad \text{(A22)}$$

$$[\mathbf{10}^{ab}\,\bar{\mathbf{10}}_{lm}] \to \left(\delta^a_l\,\delta^b_m - \delta^a_m\,\delta^b_l\right) \qquad \text{(A23)}$$

$$[\mathbf{15}^{ab}\,\bar{\mathbf{15}}_{lm}] \to \left(\delta^a_l\,\delta^b_m + \delta^a_m\,\delta^b_l\right) \qquad \text{(A24)}$$

$$\begin{aligned}[\mathbf{35}^{abc}\,\bar{\mathbf{35}}_{lmn}] \to 2\,\Big[&\left(\delta^a_l\,\delta^b_m + \delta^a_m\,\delta^b_l\right)\delta^c_n + \\ &\left(\delta^a_l\,\delta^c_m + \delta^a_m\,\delta^c_l\right)\delta^b_n + \left(\delta^c_l\,\delta^b_m + \delta^c_m\,\delta^b_l\right)\delta^a_n\Big] \qquad \text{(A25)}\end{aligned}$$

$$[\mathbf{40}^{abc}\,\bar{\mathbf{40}}_{lmn}] \to 3\left(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l\right)\delta^c_n - \frac{1}{2}\epsilon^{ijabc}\epsilon_{ijlmn} \quad \text{(A26)}$$

$$\begin{aligned}[\mathbf{70}^{ab}_c\,\bar{\mathbf{70}}^n_{lm}] \to\ & 6\left(\delta^a_l\,\delta^b_m + \delta^a_m\,\delta^b_l\right)\delta^n_c \\ & - \left(\delta^b_l\,\delta^n_m - \delta^b_m\,\delta^n_l\right)\delta^a_c + \left(\delta^a_l\,\delta^n_m + \delta^a_m\,\delta^n_l\right)\delta^b_c. \quad \text{(A27)}\end{aligned}$$

The index structures for the reducible $\mathbf{25}^{(r)}$ and $\mathbf{50}^{(r)}$ used in sec. IV B are:

$$[\mathbf{25}^{(r)\,ab}\,\bar{\mathbf{25}}^{(r)}_{lm}] \to \quad 2\,\delta^a_l\,\delta^b_m \qquad \text{(A28)}$$

$$[\mathbf{50}^{(r)\,ab}_c\,\bar{\mathbf{50}}^{(r)\,n}_{lm}] \to -4\,\delta^n_c\left[\delta^a_l\,\delta^b_m - \delta^a_m\,\delta^b_l\right]. \qquad \text{(A29)}$$

APPENDIX B: TABLES OF RESULTS

In tables XI to XIV we collect the coefficients of the mass operators contributing to the effective Yukawa couplings Y^d and Y^ℓ at $\mathcal{O}(\epsilon,\,\epsilon^2,\,\epsilon^3)$. The operators are evaluated with a factor $-i\mathcal{V}$ for each vertex and $-i\mathcal{S}/M$ for each propagator, and by dividing the result by i.

	$\mathcal{O}(\epsilon)$	$\frac{Y^\ell}{\epsilon/\sqrt{60}}$	$\frac{Y^d}{\epsilon/\sqrt{60}}$	
1)	**10**	6	1	$\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma}$
2)	**15**	0	5	
3)	**5**	-3	2	$\mathbf{\Sigma}\bar{\phi}_{\mathbf{d}}$
4)	**45**	15	10	

TABLE XI: Operators contributing to Y^ℓ and Y^d at $\mathcal{O}(\epsilon)$.

	$\mathcal{O}(\epsilon^2)$	$\frac{Y^\ell}{(\epsilon/\sqrt{60})^2}$	$\frac{Y^d}{(\epsilon/\sqrt{60})^2}$	
1)	$\mathbf{10}\ \mathbf{10}$	-36	-1	$\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma\Sigma}$
2)	$\mathbf{10}\ \mathbf{15}$	0	-25	
3)	$\mathbf{10}\ \overline{\mathbf{40}}$	0	-50	
4)	$\mathbf{15}\ \mathbf{10}$	0	-5	
5)	$\mathbf{15}\ \mathbf{15}$	0	5	
6)	$\mathbf{5}\ \mathbf{10}$	18	-2	$\mathbf{\Sigma}\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma}$
7)	$\mathbf{5}\ \mathbf{15}$	0	-10	
8)	$\mathbf{45}\ \mathbf{10}$	-90	-10	
9)	$\mathbf{45}\ \overline{\mathbf{40}}$	0	-200	
10)	$\mathbf{70}\ \mathbf{15}$	0	100	
11)	$\mathbf{5}\ \mathbf{5}$	-9	-4	$\mathbf{\Sigma\Sigma}\bar{\phi}_{\mathbf{d}}$
12)	$\mathbf{45}\ \mathbf{5}$	75	100	
13)	$\mathbf{70}\ \mathbf{5}$	-225	-200	
14)	$\mathbf{45}_\uparrow\ \mathbf{45}$	285	-60	
15)	$\mathbf{45}_\downarrow\ \mathbf{45}$	105	20	
16)	$\mathbf{5}\ \mathbf{45}$	45	-20	
17)	$\mathbf{70}\ \mathbf{45}$	225	-200	

TABLE XII: Operators contributing to Y^ℓ and Y^d at $\mathcal{O}(\epsilon^2)$.

	$\mathcal{O}(\epsilon^3)$	$\frac{Y^\ell}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d}{(\epsilon/\sqrt{60})^3}$	
1)	$\mathbf{10}\ \mathbf{10}\ \mathbf{10}$	216	1	$\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma\Sigma\Sigma}$
2)	$\mathbf{10}\ \mathbf{10}\ \mathbf{15}$	0	25	
3)	$\mathbf{10}\ \mathbf{10}\ \overline{\mathbf{40}}$	0	50	
4)	$\mathbf{10}\ \mathbf{15}\ \mathbf{10}$	0	25	
5)	$\mathbf{10}\ \mathbf{15}\ \mathbf{15}$	0	-25	
6)	$\mathbf{10}\ \overline{\mathbf{40}}\ \mathbf{10}$	0	50	
7)	$\mathbf{10}\ \overline{\mathbf{40}}_\uparrow\ \overline{\mathbf{40}}$	0	-200	
8)	$\mathbf{10}\ \overline{\mathbf{40}}_\downarrow\ \overline{\mathbf{40}}$	0	350	
9)	$\mathbf{15}\ \mathbf{10}\ \mathbf{10}$	0	5	
10)	$\mathbf{15}\ \mathbf{10}\ \mathbf{15}$	0	125	
11)	$\mathbf{15}\ \mathbf{10}\ \overline{\mathbf{40}}$	0	250	
12)	$\mathbf{15}\ \mathbf{15}\ \mathbf{10}$	0	-5	
13)	$\mathbf{15}\ \mathbf{15}\ \mathbf{15}$	0	5	
14)	$\mathbf{5}\ \mathbf{10}\ \mathbf{10}$	-108	2	$\mathbf{\Sigma}\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma\Sigma}$
15)	$\mathbf{5}\ \mathbf{10}\ \mathbf{15}$	0	50	
16)	$\mathbf{5}\ \mathbf{10}\ \overline{\mathbf{40}}$	0	100	
17)	$\mathbf{5}\ \mathbf{15}\ \mathbf{10}$	0	10	
18)	$\mathbf{5}\ \mathbf{15}\ \mathbf{15}$	0	-10	
19)	$\mathbf{45}\ \mathbf{10}\ \mathbf{10}$	540	10	
20)	$\mathbf{45}\ \mathbf{10}\ \mathbf{15}$	0	250	
21)	$\mathbf{45}\ \mathbf{10}\ \overline{\mathbf{40}}$	0	500	
22)	$\mathbf{45}\ \overline{\mathbf{40}}\ \mathbf{10}$	0	-200	
23)	$\mathbf{45}\ \overline{\mathbf{40}}_\uparrow\ \overline{\mathbf{40}}$	0	800	
24)	$\mathbf{45}\ \overline{\mathbf{40}}_\downarrow\ \overline{\mathbf{40}}$	0	-1400	
25)	$\mathbf{70}\ \mathbf{15}\ \mathbf{10}$	0	-100	
26)	$\mathbf{70}\ \mathbf{15}\ \mathbf{15}$	0	100	

TABLE XIII: Operators contributing to Y^ℓ and Y^d at $\mathcal{O}(\epsilon^3)$ (continued below).

	$\mathcal{O}(\epsilon^3)$	$\frac{Y^\ell}{(\epsilon/\sqrt{60})^3}$	$\frac{Y^d}{(\epsilon/\sqrt{60})^3}$	
27)	$\mathbf{5}\ \mathbf{5}\ \mathbf{10}$	54	4	$\mathbf{\Sigma\Sigma}\bar{\phi}_{\mathbf{d}}\mathbf{\Sigma}$
28)	$\mathbf{5}\ \mathbf{5}\ \mathbf{15}$	0	20	
29)	$\mathbf{5}\ \mathbf{45}\ \mathbf{10}$	-270	20	
30)	$\mathbf{5}\ \mathbf{45}\ \overline{\mathbf{40}}$	0	400	
31)	$\mathbf{5}\ \mathbf{70}\ \mathbf{15}$	0	-200	
32)	$\mathbf{45}\ \mathbf{5}\ \mathbf{10}$	-450	-100	
33)	$\mathbf{45}\ \mathbf{5}\ \mathbf{15}$	0	-500	
34)	$\mathbf{45}_\uparrow\ \mathbf{45}\ \mathbf{10}$	-1710	60	
35)	$\mathbf{45}_\downarrow\ \mathbf{45}\ \mathbf{10}$	-630	-20	
36)	$\mathbf{45}_\uparrow\ \mathbf{45}\ \overline{\mathbf{40}}$	0	-1200	
37)	$\mathbf{45}_\downarrow\ \mathbf{45}\ \overline{\mathbf{40}}$	0	400	
38)	$\mathbf{45}\ \overline{\mathbf{50}}\ \overline{\mathbf{40}}$	0	$-$	
39)	$\mathbf{45}\ \mathbf{70}\ \mathbf{15}$	0	1000	
40)	$\mathbf{70}\ \mathbf{5}\ \mathbf{10}$	1350	200	
41)	$\mathbf{70}\ \mathbf{5}\ \mathbf{15}$	0	1000	
42)	$\mathbf{70}\ \mathbf{45}\ \mathbf{10}$	-1350	200	
43)	$\mathbf{70}\ \mathbf{45}\ \overline{\mathbf{40}}$	0	4000	
44)	$\mathbf{70}_\uparrow\ \mathbf{70}\ \mathbf{15}$	0	-400	
45)	$\mathbf{70}_\downarrow\ \mathbf{70}\ \mathbf{15}$	0	800	
46)	$\mathbf{5}\ \mathbf{5}\ \mathbf{5}$	-27	8	$\mathbf{\Sigma\Sigma\Sigma}\bar{\phi}_{\mathbf{d}}$
47)	$\mathbf{45}\ \mathbf{5}\ \mathbf{5}$	225	-200	
48)	$\mathbf{70}\ \mathbf{5}\ \mathbf{5}$	-675	400	
49)	$\mathbf{5}\ \mathbf{45}\ \mathbf{5}$	225	-200	
50)	$\mathbf{45}_\uparrow\ \mathbf{45}\ \mathbf{5}$	1425	-600	
51)	$\mathbf{45}_\downarrow\ \mathbf{45}\ \mathbf{5}$	525	200	
52)	$\mathbf{70}\ \mathbf{45}\ \mathbf{5}$	1125	-2000	
53)	$\mathbf{5}\ \mathbf{70}\ \mathbf{5}$	-675	400	
54)	$\mathbf{45}\ \mathbf{70}\ \mathbf{5}$	1125	-2000	
55)	$\mathbf{70}_\uparrow\ \mathbf{70}\ \mathbf{5}$	-4725	800	
56)	$\mathbf{70}_\downarrow\ \mathbf{70}\ \mathbf{5}$	-675	-1600	
57)	$\mathbf{5}\ \mathbf{5}\ \mathbf{45}$	135	40	
58)	$\mathbf{45}\ \mathbf{5}\ \mathbf{45}$	-1125	-1000	
59)	$\mathbf{70}\ \mathbf{5}\ \mathbf{45}$	3375	2000	
60)	$\mathbf{5}\ \mathbf{45}_\uparrow\ \mathbf{45}$	855	120	
61)	$\mathbf{5}\ \mathbf{45}_\downarrow\ \mathbf{45}$	315	-40	
62)	$\mathbf{45}_\uparrow\ \mathbf{45}_\uparrow\ \mathbf{45}$	5415	360	
63)	$\mathbf{45}_\uparrow\ \mathbf{45}_\downarrow\ \mathbf{45}$	1995	-120	
64)	$\mathbf{45}_\downarrow\ \mathbf{45}_\uparrow\ \mathbf{45}$	1995	-120	
65)	$\mathbf{45}_\downarrow\ \mathbf{45}_\downarrow\ \mathbf{45}$	735	40	
66)	$\mathbf{70}\ \mathbf{45}_\uparrow\ \mathbf{45}$	4275	1200	
67)	$\mathbf{70}\ \mathbf{45}_\downarrow\ \mathbf{45}$	1575	-400	
68)	$\mathbf{5}\ \mathbf{70}\ \mathbf{45}$	675	400	
69)	$\mathbf{45}\ \mathbf{70}\ \mathbf{45}$	-1125	-2000	
70)	$\mathbf{70}_\uparrow\ \mathbf{70}\ \mathbf{45}$	4725	800	
71)	$\mathbf{70}_\downarrow\ \mathbf{70}\ \overline{\mathbf{45}}$	675	-1600	

TABLE XIV: Operators contributing to Y^ℓ and Y^d at $\mathcal{O}(\epsilon^3)$.

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[26] The number quoted correspond to $\tan\beta = 1.3$. For larger $\tan\beta$ and neglecting threshold corrections, a smaller value $m_b/m_\tau \approx 0.73 \pm 0.03$ is found [1].
[27] For larger $\tan\beta$ ref. [1] quotes $3m_s/m_\mu \approx 0.69 \pm 0.08$ while $m_d/3m_e$ is practically unaffected.
[28] While there are theoretical frameworks in which Yukawa couplings obey to some principle of universality, like e.g. superstring-inspired models [18], SUSY-GUT models for gauge-Yukawa unification [19], or models for gauge-Higgs unification in higher dimensions [20], the fundamental Yukawa couplings of our model involve rather large vectorlike representations that cannot be accommodated easily in these scenarios.
[29] At $\mathcal{O}(\epsilon^3)$ we omit a few operators induced by representation with dimension > 100. We also omit one operator induced by the **50** since deriving its group index structure is rather awkward.
[30] The fact that the sum of $O(\epsilon; \mathbf{10})$ and $O(\epsilon; \mathbf{15})$ preserves b-τ unification was already noted in ref. [17].